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CHIRAL ANOMALY AND NUCLEON PROPERTIES IN THE NAMBU–JONA-LASINIO MODEL WITH VECTOR MESONS

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ABSTRACT

We consider the extended SU(3) Nambu and Jona-Lasinio model with explicit vector couplings in the presence of external fields. We study the chiral anomaly in this model and its implications on the properties of the nucleon described as a chiral soliton of three valence quarks bounded in mesonic background fields. For the model to reproduce the QCD anomaly it is necessary to subtract suitable local and polynomial counterterms in the external and dynamical vector and axial-vector fields. We compute the counterterms explicitly in a vector gauge invariant regularization, and obtain modifications to the total effective action and vector and axial currents. We study the numerical influence of those counterterms in the two flavour version of model with dynamical σ, π, ρ, A and ω mesons. We find that, for time independent hedgehog configurations, the numerical effects in the nucleon mass, the isoscalar nucleon radius and the axial coupling constant are negligibly small.

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1. Introduction

The QCD chiral anomaly in its vector gauge invariant form [1] represents an important constraint for any effective low energy model of hadronic interactions. It is responsible for the $\pi^0 \to 2\gamma$ decay, provides evidence that the number of colours N_c is equal to three and is an exact one loop result not subjected to renormalization [2]. From the point of view of the effective Lagrangian and depending upon the degrees of freedom involved, there are at least two ways in which the chiral anomaly can be introduced. In a pure mesonic theory one can add the vector gauged Wess-Zumino term [3,4] to the action which by definition saturates the QCD anomaly equation. This type of construction and similar ones have been extensively used to calculate abnormal parity mesonic processes [5,6] or to provide short range repulsion in the construction of topological chiral soliton models of the Skyrme type [7]. Proceeding in this way there is in principle an ambiguity concerning the possible abnormal parity non-anomalous terms which do not contribute to the anomaly equation. One should say, however, that the ambiguity is completely removed if only pions and external fields are considered, but remains if other particles like e.g. heavy vector resonances are included. This allows to fix the corresponding coefficients in the chiral Lagrangian to fulfill some desired properties motivated by phenomenology and not directly derivable from the chiral anomaly [6].

In a relativistic chiral quark model the situation is slightly different. The non invariance of the functional integration measure or equivalently of the fermion determinant under chiral transformations [8,9] guarantees the onset of a chiral anomaly in the effective theory whose particular form depends on the symmetries preserved by the regulator. Actually, it is by no means clear that the resulting anomaly coincides with the QCD anomaly. In such a situation, the question arises whether the quark model can be redefined in such a way that the correct QCD anomaly can be reproduced.

A prototype of a chiral quark model for hadronic structure is represented by the Nambu–Jona-Lasinio (NJL) model [10]. This model has been studied intensively in the vacuum sector, the meson sector and the baryon sector [11]. It is a pure quark model which incorporates explicit, spontaneous and anomalous chiral symmetry breaking, describes mesons as quark-antiquark pairs and baryons as solitons of three bound quarks or alternatively as quark-diquark bound states. However, it does not include confinement and it is not renormalizable requiring the use of a low energy cut-off.

The generalization of the model to include vector and axial-vector couplings has also been studied in much detail either in a bosonized version [12,13,14,15,16,17] or in a pure quark language [18,19,20,21] since it allows to describe more mesonic phenomenology and provides a realization of vector meson dominance through current field identities [22].

Actually, we have shown in a recent paper [23] that the Nambu and Jona-Lasinio model with scalar, pseudoscalar, vector and axial mesons as it is most often used does not reproduce the anomaly of QCD, the correct result being reproduced if the vector couplings are set equal to zero. This result is perfectly consistent with vector gauge invariance, i.e. only the divergence of the axial current is modified with respect to the QCD result whereas the vector current is still conserved. Curiously, the $\pi^0 \to 2\gamma$ decay amplitude remained unchanged, but other anomalous amplitudes involving more than one pion, as for instance $\gamma \to 3\pi$ deviated from the current algebra result by $\sim 20\%$ for typical values of the parameters. This is an unpleasant feature because questions the meaning of previous calculations involving anomalous vertices, both in the meson [24] and the soliton sector [25,26,27,28,29,30,31] based on actions with vector mesons not containing the proper anomalous structure [1]. The problem arises whether the non fulfillment of the QCD chiral anomaly is a problem of describing vector mesons through vector couplings or whether it can be mended by addition of legitimate counterterms in order to reproduce the QCD anomaly in its Bardeen form. In any case, it is desirable to know the possible implications not only for mesonic decays but also at the nucleon level. To overcome this problem Bijnens and Prades [32] have suggested, following elegant mathematical arguments already given by Bardeen and Zumino [33] and pursued by others (see e.g. [5] and references included in [32]), that indeed such counterterms exist and that they are unique if CP invariance is invoked. This represents

another independent non trivial solution to the anomaly equation not considered explicitly in our previous work [23]. The choice between these two solutions depends on our desire to include explicit vector and axial couplings in the starting NJL Lagrangian and at the same time being able to reproduce the QCD chiral anomaly. However, it should be stressed that the subtraction involves not only the external fields but also the dynamical vector fields. From the point of view of perturbation theory this corresponds to change an infinite number of diagrams in contrast to the usual external field subtraction procedure. In this respect this is a very particular feature of the NJL model and its singularity structure. As a consequence vector meson dominance in the usual sense is lost. This means that a consistent Lagrangian incorporating both the QCD anomaly and complete vector meson dominance cannot be constructed, at least in the vector field representation of vector mesons.

In the approach of ref. [32] the calculation of the counterterms requires an explicit knowledge of the vector and axial currents. In addition, the formulas given by those authors are rather compact and of low practical utility in the present context. In the present paper we adhere to the point of view of ref. [32] although propose a somewhat different methodology which only relies on the knowledge of the anomaly itself. In fact we trace back the origin of the ambiguity to the natural regularization suggested by the bosonization procedure widely used for performing low energy expansions, determination of mesonic vertex functions and description of baryons as solitons. We derive the modifications of the effective action and the vector and axial currents and obtain the leading large N_c corrections to the Current-Field Identities [22]. We also particularize the resulting formulas for the the specific two flavour case with σ, π, ρ, A and ω mesons and compute the numerical modification of the nucleon energy, the isoscalar nucleon radius and the axial coupling constant in the solitonic picture of baryons. As a byproduct we also compute the leading low energy chiral invariant, i.e. non anomalous, contribution to the abnormal parity action of vector mesons in the presence of external fields. We also study the form of the possible CP violating currents.

The paper is organized as follows. In section 2 we review the model to fix the notation used along the paper. In section 3 we derive the general Ward identities from the NJL generating functional, and in particular, the form of the chiral anomaly which appears in the model. In section 4 the regularization in Minkowski space is described. In section 5 we describe the counterterms which allow to reproduce the QCD anomaly as well as their influence on the effective action and vector and axial currents. The constructive method used to derive the counterterms is described in detail in section 6. Section 7 is devoted to the study of the particular two flavour case with σ, π, ρ, A and ω mesons. In section 8 we present our numerical results for nucleon observables. Section 9 deals with the effective low energy abnormal parity but not anomalous action of vector mesons in the presence of external currents. Finally, in section 10 we summarize our results and present our conclusions.

2. The NJL Model and Bosonization Revisited

Our starting point is the Nambu–Jona-Lasinio Lagrangian in Minkwoski space [11]

$$\begin{aligned}\mathcal{L}_{\rm NJL} = \bar{q}(i\partial\!\!\!/ - \hat{M}_0)q + \frac{G_S}{2}\sum_{a=0}^{N_f^2-1}\left((\bar{q}\lambda_a q)^2 + (\bar{q}\lambda_a i\gamma_5 q)^2\right) \\ - \frac{G_V}{2}\sum_{a=0}^{N_f^2-1}\left((\bar{q}\lambda_a \gamma_\mu q)^2 + (\bar{q}\lambda_a \gamma_\mu\gamma_5 q)^2\right)\end{aligned} \tag{2.1}$$

where $q = (u, d, s, \ldots)$ represents a quark spinor with N_c colours and N_f flavours. The λ's represent the Gell-Mann matrices of the $U(N_f)$ group (see Appendix A) and $\hat{M}_0 = \mathrm{diag}(m_u, m_d, m_s, ...)$ stands for the current quark mass matrix. In the limiting case of vanishing current quark masses the NJL-action is invariant under the global $U(N_f)_R \otimes U(N_f)_L$ group of transformations (see e.g. Appendix A). The corresponding vector and axial currents are given by

$$J^V_{\mu a}(x) = \frac{1}{2}\bar{q}(x)\gamma_\mu\lambda_a q(x); \qquad J^A_{\mu a}(x) = \frac{1}{2}\bar{q}(x)\gamma_\mu\gamma_5\lambda_a q(x); \tag{2.2}$$

respectively. We will not consider the effects of a $U_A(1)$ breaking term as done in [35] since they are not relevant for what follows. In order not to overload the paper with notation we will always work in Minkowski space and will never specify the Wick rotation explicitly. In fact well defined results can be obtained by using the customary replacement $\hat{M}_0 \to \hat{M}_0 - i\epsilon$. Nevertheless all our results can be equivalently obtained from a Euclidean analytical continuation following the conventions of appendix B.

The vacuum to vacuum transition amplitude in the presence of external bosonic (s, p, v, a) and fermionic $(\eta, \bar{\eta})$ fields of the NJL Lagrangian can be written as a path integral as

$$\begin{aligned} Z[s,p,v,a,\eta,\bar{\eta}] &= \langle 0|\mathrm{T}\exp\Big\{i\int d^4x\Big[\bar{q}\Big(\not{v} + \not{a}\gamma_5 - (s+i\gamma_5 p)\Big)q + \bar{\eta}q + \bar{q}\eta\Big]\Big\}|0\rangle \\ &= \int D\bar{q}Dq\exp\Big\{i\int d^4x\Big[\mathcal{L}_{\mathrm{NJL}} + \bar{q}\Big(\not{v} + \not{a}\gamma_5 - (s+i\gamma_5 p)\Big)q + \bar{\eta}q + \bar{q}\eta\Big]\Big\} \end{aligned} \tag{2.3}$$

Following the standard procedure [36] it is convenient to introduce auxiliary bosonic fields so that one gets the equivalent generating functional

$$Z[s,p,v,a,\eta,\bar{\eta}] = \int D\bar{q}DqDSDPDVDA\exp\Big\{i\int d^4x\mathcal{L}_{\mathrm{sm}}(x)\Big\} \tag{2.4}$$

where the semi-bosonized Lagrangian reads

$$\mathcal{L}_{\mathrm{sm}} = \mathcal{L}_{\mathrm{int}} + \mathcal{L}^F_{\mathrm{ext}} + \mathcal{L}^B_{\mathrm{ext}} + \mathcal{L}_m + \mathcal{L}_{M_0} \tag{2.5}$$

and

$$\begin{aligned} \mathcal{L}_{\mathrm{int}} &= \bar{q}\Big(\not{V} + \not{A}\gamma_5 - (S+i\gamma_5 P)\Big)q = -\bar{q}M_{\mathrm{int}}q \\ \mathcal{L}^B_{\mathrm{ext}} &= \bar{q}\Big(i\not{\partial} + \not{v} + \not{a}\gamma_5 - (s+i\gamma_5 p)\Big)q = \bar{q}i\mathbf{D}_{\mathrm{ext}}q \\ \mathcal{L}^F_{\mathrm{ext}} &= \bar{\eta}q + \bar{q}\eta \\ \mathcal{L}_m &= -\frac{1}{4G_S}\,\mathrm{tr}\,(S^2+P^2) + \frac{1}{4G_V}\,\mathrm{tr}\,(V_\mu V^\mu + A_\mu A^\mu) \\ \mathcal{L}_{M_0} &= -\bar{q}\hat{M}_0 q \end{aligned} \tag{2.6}$$

Here (S, P, V, A) are dynamical internal bosonic fields, whereas (s, p, v, a) and $(\eta, \bar{\eta})$ represent external bosonic and fermionic fields respectively. The bosonic fields are all of them expanded in terms of the λ flavour matrices (See Appendix A). Notice also that for the path integral in the bosonic fields to be well defined in Minkowski space we must use the prescription $\frac{1}{G_S} \to \frac{1}{G_S} - i\epsilon$ and $\frac{1}{G_V} \to \frac{1}{G_V} - i\epsilon$. Finally, if fermions are formally integrated out one obtains

$$Z[s,p,v,a,\eta,\bar{\eta}] = \int DM_{\mathrm{int}}\mathrm{Det}(i\mathbf{D})\exp\Big\{-i\langle\bar{\eta},(i\mathbf{D})^{-1}\eta\rangle\Big\}\exp\Big\{i\int d^4x\mathcal{L}_m\Big\} \tag{2.7}$$

where the bosonic integration measure $DM_{\mathrm{int}} = DSDPDVDA$ and the usual notation

$$\langle\bar{\eta},(i\mathbf{D})^{-1}\eta\rangle = \int\int d^4x d^4x'\bar{\eta}(x)\langle x|(i\mathbf{D}+i\epsilon)^{-1}|x'\rangle\eta(x') \tag{2.8}$$

have been introduced. Concerning the previous manipulations and anticipating the forthcoming sections, a few remarks should be made. The path integral above is a highly singular object and defines a whole family of effective physical theories depending on the particular prescriptions employed to give it a meaning. This includes in particular the regularization procedure and the symmetries respected by it. In the absence of a real derivation from QCD of the NJL model, one can only hope that such a freedom can be used to reproduce as many known features of the underlying fundamental theory as possible. This set of possible choices is, however, not completely arbitrary. It only reflects the ambiguity arising in the regularization of a

certain ultraviolet divergent Feynman diagram, namely the fact that in a loop graph the divergent piece is a polynomial in the external incoming momenta and also that only diagrams with less than five legs in four space-time dimensions are divergent. The coefficients of such a polynomial depend on the regularization. In practice, this means that depending on the regularization one can always subtract local and polynomial counterterms in the fields to the action. In this paper we are interested in the NJL model as an effective theory of QCD and in the possible counterterms that allow to reproduce the QCD chiral anomaly for a given regularization prescription.

It should be kept in mind that, strictly speaking, the bosonization procedure is not obligatory. Nevertheless it will prove highly convenient in what follows since it reorders the diagrammatic expansion in a convenient way, so that N_c counting rules become self-evident. More important, it is the only known device to describe in practice baryons as chiral solitons. Nevertheless, the bosonization may appear to impose some conditions on the theory, since it treats classes of infinite graphs on the same footing. This point will be discussed later in more detail.

3. Ward Identities

For the moment we will ignore the regularization and proceed formally. The resulting expressions will only acquire a precise meaning when the mathematical objects involved are provided with a suitable regularization. For simplicity we will consider the chiral limit, i.e. we set $\hat{M}_0 = 0$, since their influence is expected to be small in the present context.

To obtain the form of Ward identities we decompose first the Dirac operator into an internal plus external field contribution both transforming homogeneously under local chiral transformations, i.e. we define

$$i\mathbf{D} = i\mathbf{D}_{\text{ext}} - M_{\text{int}} \tag{3.1}$$

and

$$\mathcal{S} = s + S; \qquad \mathcal{P} = p + P; \qquad \mathcal{V} = v + V; \qquad \mathcal{A} = a + A \tag{3.2}$$

A local chiral rotation of the external fields induces the change $\mathbf{D}_{\text{ext}} \to \mathbf{D}^g_{\text{ext}}$ (see Appendix A for details and conventions). Let us call $Z^g[s,p,v,a,\eta,\bar\eta] = Z[s^g,p^g,v^g,a^g,\eta^g,\bar\eta^g]$ the generating functional so transformed. We make a change of variables in the dynamical bosonic fields $(S,P,V,A) \to (S^g,P^g,V^g,A^g)$ which in the infinitesimal case reads

$$\begin{aligned}
\delta V^\mu &= i[\epsilon_V, V^\mu] + i[\epsilon_A, A^\mu] \\
\delta A^\mu &= i[\epsilon_V, A^\mu] + i[\epsilon_A, V^\mu] \\
\delta S &= i[\epsilon_V, S] + \{\epsilon_A, P\} \\
\delta P &= i[\epsilon_V, P] - \{\epsilon_A, S\}
\end{aligned} \tag{3.3}$$

Notice that the transformation is local but homogeneous for all the internal fields, i.e. no derivative terms appear. This is required if $\mathcal{V},\mathcal{A}$ and v,a have to transform non-homogeneously. The bosonic measure is invariant under these local homogeneous chiral transformations $DSDPDVDA = DS^gDP^gDV^gDA^g$ and also the bosonic mass terms $\mathcal{L}^g_m = \mathcal{L}_m$. This property allows to freely choose whether the bosonic fields are transformed or not, i.e. whether we take $i\mathbf{D}^g = i\mathbf{D}^g_{\text{ext}} - M^g_{\text{int}}$ or $i\mathbf{D}^g = i\mathbf{D}^g_{\text{ext}} - M_{\text{int}}$ respectively. We have

$$Z^g[s,p,v,a,\eta,\bar\eta] = \int D\bar q Dq DM_{\text{int}} \exp\Big\{ i \int d^4x \Big[\bar q i\mathbf{D}^g q + \mathcal{L}_m + \bar q \eta^g + \bar\eta^g q \Big] \Big\} \tag{3.4}$$

Integrating out the fermions we get

$$Z^g[s,p,v,a,\eta,\bar\eta] = \int DM_{\text{int}} \text{Det}(i\mathbf{D}^g) \exp\Big\{ -i\langle \bar\eta, (i\mathbf{D})^{-1}\eta\rangle \Big\} \exp\Big\{ i\int d^4x \mathcal{L}_m \Big\} \tag{3.5}$$

Note that the term $\langle \bar\eta, (i\mathbf{D})^{-1}\eta\rangle$ given by eq.(2.8) is invariant under the local chiral transformations specified above. For infinitesimal transformations we have *

$$\delta Z[s,p,v,a,\eta,\bar\eta] = \int DM_{\rm int}\delta\Big(\text{Sp}\ \log(i\mathbf{D})\Big)\text{Det}(i\mathbf{D})\exp\Big\{-i\langle\bar\eta,(i\mathbf{D})^{-1}\eta\rangle\Big\}\exp\Big\{i\int d^4x\mathcal{L}_m\Big\} \tag{3.6}$$

Decomposing the variation into its vector and axial parts as follows

$$\delta\Big(\text{Sp}\ \log(i\mathbf{D})\Big) = 2i\sum_{a=0}^{N_F^2-1}\int d^4x[\epsilon_V^a(x)\mathcal{A}_V^a(x)+\epsilon_A^a(x)\mathcal{A}_A^a(x)] \tag{3.7}$$

and compare with the corresponding variation before integrating out the quarks we get the following identities

$$\begin{aligned}\partial^\mu J^V_{\mu a} &= -\mathcal{A}_V^a(x)+\\ &+f_{abc}\Big[J^V_{\mu b}v_c^\mu+J^A_{\mu b}a_c^\mu\Big]-\frac{1}{2}f_{abc}\Big[\bar q\lambda_b q s_c+\bar q i\lambda_b\gamma_5 q p_c\Big]+\frac{i}{2}\Big[\bar q\lambda_a\eta-\bar\eta\lambda_a q\Big]\\ \partial^\mu J^A_{\mu a} &= -\mathcal{A}_A^a(x)+\\ &+f_{abc}\Big[J^V_{\mu b}a_c^\mu+J^A_{\mu b}v_c^\mu\Big]+\frac{1}{2}d_{abc}\Big[\bar q\lambda_b q s_c-\bar q i\lambda_b\gamma_5 q p_c\Big]-\frac{1}{2}\Big[\bar q i\lambda_a\gamma_5\eta+\bar\eta i\lambda_a\gamma_5 q\Big]\end{aligned} \tag{3.8}$$

This identity is valid under the path integral weighted with the full semi-bosonized action given by eq. (2.5). Moreover, it is clear that since we always obtain the Ward identities by functional differentiation of an effective action, the Wess-Zumino consistency conditions [3] are automatically satisfied.

4. Effective Action and Regularization

The determinant of the Dirac operator is an ultraviolet divergent object. Hence we have to introduce some regularization. In addition, since the model is non-renormalizable the corresponding cut-off has to remain finite, at least for the divergent pieces. This is a clear theoretical uncertainty in the model which has very often been neglected besides few exceptions. Nevertheless, some constraints can be imposed on the basis of symmetries. For the benefit of the reader we will make some digression here about the choice of regularization prescription at the expense of overlapping with previous works (see specially ref.[29]).

At a formal level the determinant of the Dirac operator depends on the sum of the internal plus external fields. This circumstance is ultimately the reason for the realization of vector meson dominance through Current-Field Identities in the model. Naively, one might expect that after regularization the result depends on the sum too. This, however, does not necessarily have to be the case [32]. For instance, one may first formally expand the effective action in powers of the fields and apply a regularization prescription afterwards. Proceeding in this way one has the freedom to regularize each vertex separately in a way that additivity is not fulfilled. One can add to the Lagrangian the most general local counterterm of at most mass dimension four which does not depend on the sum of internal plus external fields as follows

$$\log\text{Det}(i\mathbf{D}) := \log\overline{\text{Det}}(i\mathbf{D})+i\Delta\Gamma[v,a;V,A] \tag{4.1}$$

where the bar stands for a vector additive regularized fermion determinant and $\Delta\Gamma$ are the counterterms. Such a decomposition is convenient within the chiral soliton approach to baryonic structure. In the NJL model, the regularized determinant represents the contribution of the polarized Dirac sea to baryonic observables and is usually evaluated as a regularized sum of eigenvalues, thus conserving additivity. This corresponds to the $\log\overline{\text{Det}}(i\mathbf{D})$ piece. In the next section we will discuss the constraints on the counterterms $\Delta\Gamma$. In the remainder of this section we give our precise definition of $\log\overline{\text{Det}}(i\mathbf{D})$.

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* The total trace Sp is made out of the space-time trace, the colour trace, the Dirac spinor trace tr_γ and the flavour trace tr.

The additive contribution to the effective action can be separated into a γ_5-odd and γ_5-even part. It is convenient to introduce the operator

$$\mathbf{D}_5[\mathcal{S},\mathcal{P},\mathcal{V},\mathcal{A}] = \gamma_5 \mathbf{D}[\mathcal{S},-\mathcal{P},\mathcal{V},-\mathcal{A}]\gamma_5 = -\mathbf{D}[-\mathcal{S},\mathcal{P},\mathcal{V},-\mathcal{A}] \tag{4.2}$$

In fact, $\mathbf{D}_5$ corresponds to rotate $\mathbf{D}$ to Euclidean space, take the hermitean conjugate and rotate back to Minkowski space (see Appendix B). This definition allows to separate the action into a γ_5-even part (normal pseudoparity) and a γ_5-odd part (abnormal pseudoparity). The former can be regularized in a chiral gauge invariant manner by means of the Pauli-Villars scheme [37]

$$\begin{aligned}
\log \overline{\mathrm{Det}}|_{\mathrm{even}} &= \frac{1}{2}\Big[\mathrm{Splog}(i\mathbf{D}+i\epsilon) + \mathrm{Splog}(i\mathbf{D}_5+i\epsilon)\Big] \\
&= \frac{1}{4}\Big[\mathrm{Splog}(\mathbf{D}\mathbf{D}_5+i\epsilon) + \mathrm{Splog}(\mathbf{D}_5\mathbf{D}+i\epsilon)\Big] \\
&\to \frac{1}{4}\mathrm{Sp}\sum c_i\Big[\log(\mathbf{D}\mathbf{D}_5+\Lambda_i^2+i\epsilon) + \log(\mathbf{D}_5\mathbf{D}+\Lambda_i^2+i\epsilon)\Big]
\end{aligned} \tag{4.3}$$

where the Pauli-Villars regulators fulfill $c_0 = 1$, $\Lambda_0 = 0$ and $\sum_i c_i = 0$, $\sum_i c_i\Lambda_i^2 = 0$. For the γ_5-odd part we formally have

$$\begin{aligned}
\log \overline{\mathrm{Det}}|_{\mathrm{odd}} &= \frac{1}{2}\Big[\mathrm{Splog}(i\mathbf{D}+i\epsilon) - \mathrm{Splog}(i\mathbf{D}_5+i\epsilon)\Big] \\
&= \frac{1}{4}\Big[\mathrm{Splog}(\mathbf{D}^2-i\epsilon) - \mathrm{Splog}(\mathbf{D}_5^2-i\epsilon)\Big]
\end{aligned} \tag{4.4}$$

The main difference to the γ_5-even part is that the sum of the eigenvalues implied in eq. (4.4) is conditionally convergent without the need of an explicit cut-off. The result is however unambiguous if one further imposes reproducing the additive anomaly $G_A[\mathcal{V},\mathcal{A}]$ (see eq. (5.4)). This is a consequence of the absence of vector-gauge invariant fourth order terms of abnormal pseudoparity. A practical formal expression suitable for a heat-kernel expansion is given by

$$\log \overline{\mathrm{Det}}|_{\mathrm{odd}} = -\frac{1}{4}\int_0^\infty \frac{d\alpha}{\alpha}\mathrm{Sp}\Big[e^{-i\alpha(\mathbf{D}^2-i\epsilon)} - e^{-i\alpha(\mathbf{D}_5^2-i\epsilon)}\Big] \tag{4.5}$$

For this expression to be well defined one must take the limit $\epsilon \to 0^+$ at the end of the calculation.

Finally, it should be mentioned that our separation into odd and even parts implies in itself a regularization procedure. Instead one might consider a vector additive regularization applied directly to the determinant of the Dirac operator, considering the operator $\mathbf{D}$ only. The corresponding action so regularized would differ in general from ours by vector gauge invariant, additive and chirally breaking polynomial counterterms whose coefficients may depend on the regularization. This feature is due to the fact that regularization and separation into odd and even parts are not commuting operations and reflects once more the arbitrariness in the definition of the fermion determinant. Thus, if such a kind of regularization procedure were used these additional counterterms should have to be subtracted.

5. Chiral Anomaly, Counterterms and Currents in Minkowski space

Under a local chiral rotation the variation of the Dirac determinant can be separated into two contributions. Due to the transformation properties

$$\begin{aligned}
&\delta(\mathbf{D}\mathbf{D}_5) = i[\epsilon_V - \epsilon_A\gamma_5, \mathbf{D}\mathbf{D}_5]\,; \qquad \delta(\mathbf{D}_5\mathbf{D}) = i[\epsilon_V + \epsilon_A\gamma_5, \mathbf{D}_5\mathbf{D}] \\
&\delta(\mathbf{D}^2) = +i[\epsilon_V, \mathbf{D}^2] - i\{\{\epsilon_A\gamma_5, \mathbf{D}\}, \mathbf{D}\}\,; \qquad \delta(\mathbf{D}_5^2) = +i[\epsilon_V, \mathbf{D}_5^2] + i\{\{\epsilon_A\gamma_5, \mathbf{D}_5\}, \mathbf{D}_5\}
\end{aligned} \tag{5.1}$$

and using the trace cyclic property, valid under regularization, we get

$$\delta \log \overline{\mathrm{Det}}|_{\mathrm{even}} = 0 \tag{5.2}$$

for the even piece whereas for the odd piece we obtain

$$\delta \log \overline{\mathrm{Det}}|_{\mathrm{odd}} = -i \lim_{\eta \to 0^+} \mathrm{Sp}\Big[\epsilon_A \gamma_5 \Big(e^{-i\eta \mathbf{D}^2} + e^{-i\eta \mathbf{D}_5^2} \Big)\Big] \tag{5.3}$$

Straightforward calculation using the usual heat kernel method [38] yields the following result for the infinitesimal change of the regularized determinant under local chiral transformations

$$\delta \log \overline{\mathrm{Det}}(i\mathbf{D}) = +i \int d^4x \, \mathrm{tr}\,[\epsilon_A(x) \mathcal{A}_A(x)] = i G_A[\mathcal{V}, \mathcal{A}] \tag{5.4}$$

where

$$\begin{aligned} \mathcal{A}_A(x) = \frac{N_c}{4\pi^2} \epsilon_{\mu\nu\alpha\beta} \Big\{ & \frac{1}{4} \mathcal{F}^{\mu\nu} \mathcal{F}^{\alpha\beta} + \frac{1}{3} \mathcal{A}^\mu \mathcal{A}^\nu \mathcal{A}^\alpha \mathcal{A}^\beta + \\ & + \frac{i}{6} \{\mathcal{F}^{\mu\nu}, \mathcal{A}^\alpha \mathcal{A}^\beta\} + \frac{2i}{3} \mathcal{A}^\mu \mathcal{F}^{\nu\alpha} \mathcal{A}^\beta + \frac{1}{3} [\mathcal{D}^\mu, \mathcal{A}^\nu][\mathcal{D}^\alpha, \mathcal{A}^\beta] \Big\} \end{aligned} \tag{5.5}$$

with

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{V}_\nu - \partial_\nu \mathcal{V}_\mu - i[\mathcal{V}_\mu, \mathcal{V}_\nu] = i[\mathcal{D}_\mu, \mathcal{D}_\mu] \tag{5.6}$$

Notice that the fields appearing in the expression of the anomaly are the sum of internal plus external fields. The QCD anomaly in its Bardeen form depends on the external fields only and corresponds to put in eq. (5.5) $\mathcal{A} = a$ and $\mathcal{V} = v$ or correspondingly $V = 0$ and $A = 0$. Hence the NJL model reproduces the proper anomaly * if $G_V = 0$ and $\Delta\Gamma = 0$ defined by eq. (4.1). This is the solution found in our previous paper [23]. However, this is not the only solution. To keep the line of reasoning straight we anticipate the result to be derived in the next section. Another solution (unique up to CP violating terms) is given by $G_V \neq 0$ and

$$\begin{aligned} \Delta\Gamma = -\frac{iN_c}{24\pi^2} \int \mathrm{tr}\, \Big(& 6ia\{F, V\} + 3iF[A, V] + 4a^3 V + a^2[A, V] + 2a\{A^2, V\} + 4aVaA \\ & + 4aV^3 + 2iDa[a, A] + iDA[a, A] + 3iDV[a, V] + 2iDV[A, V] - VA^3 - 3V^3 A \Big) \end{aligned} \tag{5.7}$$

where for convenience we have used notation of differential forms with the following 1-forms

$$V = V_\mu dx^\mu; \qquad A = A_\mu dx^\mu; \qquad v = v_\mu dx^\mu; \qquad a = a_\mu dx^\mu \tag{5.8}$$

and 2-forms

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu; \qquad DV = \frac{1}{2} (DV)_{\mu\nu} dx^\mu dx^\nu; \qquad \text{etc.} \tag{5.9}$$

and

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu v_\nu - \partial_\nu v_\mu - i[v_\mu, v_\nu] = i[D_\mu, D_\nu] \\ (DA)_{\mu\nu} &= [D_\mu, A_\nu] - [D_\nu, A_\mu] \\ (DV)_{\mu\nu} &= [D_\mu, V_\nu] - [D_\nu, V_\mu] \\ (Da)_{\mu\nu} &= [D_\mu, a_\nu] - [D_\nu, a_\mu] \end{aligned} \tag{5.10}$$

and $dx^\mu dx^\nu dx^\alpha dx^\beta = d^4x \epsilon^{\mu\nu\alpha\beta}$. These terms satisfy that

$$\begin{aligned} \delta\Big(-i \log \overline{\mathrm{Det}}(i\mathbf{D}) + \Delta\Gamma \Big) &= G_A[v, a] \\ &= \frac{N_c}{4\pi^2} \int \mathrm{tr}\, \Big\{ \epsilon_A \Big[F^2 + \frac{1}{3} a^4 + \frac{i}{3} \{F, a^2\} + \frac{4i}{3} aFa + \frac{1}{3} (Da)^2 \Big] \Big\} \end{aligned} \tag{5.11}$$

where the fermionic contribution to the effective action is regularized in a vector gauge invariant manner. We point out that the former eq. (5.11) corresponds to eq. (5.5) in the particular case $V = 0$ and $A = 0$.

* There is a sign difference with the work of Bardeen [1] due to the different convention for the Levi-Civita tensor $\epsilon_{0123}(\mathrm{Bardeen}) = +1$ whereas we have $\epsilon^{0123} = -\epsilon_{0123} = +1$.

Notice also that the counterterms do not depend on the additive combination $\mathcal{V} = v + V$ and $\mathcal{A} = a + A$. Hence we will have corrections to the usual Current-Field Identities. Furthermore, the counterterms are also written in a manifestly vector gauge invariant fashion.

The counterterms can be classified according to the number of external fields. For our purposes only the zeroth order (modification of the action $\Delta\Gamma_0$) and first order (modification of the currents ΔJ_V and ΔJ_A) will be needed. They are

$$\begin{aligned}\Delta\Gamma_0 &= -\frac{iN_c}{24\pi^2}\int \mathrm{tr}\Big[2idV[A,V] - VA^3 - 3V^3A\Big]\\ \Delta\Gamma_1 &= -\frac{iN_c}{24\pi^2}\int \mathrm{tr}\Big[v\Big(3i\{dA,V\} - 3i\{dV,A\} + 4VAV - 2\{A,V^2\}\Big)+\\ &\qquad + a\Big(3i\{dV,V\} + i\{dA,A\} + 4V^3 + 2\{A^2,V\}\Big)\Big] =\\ &= \int \mathrm{tr}\Big(v\Delta J_V + a\Delta J_A\Big)\end{aligned} \tag{5.12}$$

The first order term can be also expressed in another form if use is made of the self-consistent equations of motion. The total action in Minkowski space can be written as

$$W = \overline{W}[s+S, p+P, v+V, a+A] + \Delta\Gamma[v,a;V,A] + W_m[S,P,V,A] - \langle\bar\eta, (i\mathbf{D})^{-1}\eta\rangle \tag{5.13}$$

where we have emphasized the fact that in our particular regularization the fermion determinant $\overline{W} = -i\log\overline{\mathrm{Det}}(i\mathbf{D})$ depends on the additive combinations $s+S$, etc. The term in the external fermionic fields depends also on these additive combinations.

The former expressions can be brought into a more appealing form by considering a saddle point approximation of the action (which in this model becomes exact in the large N_c limit). In effect, if we minimize the total action with respect to the dynamical fields V and A (the minimization with respect to S and P is not relevant in what follows) in the presence of external fields we find that under the path integral

$$\frac{\delta\overline{W}}{\delta V} + \frac{\delta\Delta\Gamma}{\delta V} + \frac{1}{2G_V}V - \frac{\delta}{\delta V}\langle\bar\eta, (i\mathbf{D})^{-1}\eta\rangle = 0 \tag{5.14}$$

and similarly for A. On the other hand, the total currents are

$$\begin{aligned}J_V &= \frac{\delta\overline{W}}{\delta v} + \frac{\delta\Delta\Gamma}{\delta v} - \frac{\delta}{\delta v}\langle\bar\eta, (i\mathbf{D})^{-1}\eta\rangle\\ &= \frac{\delta\overline{W}}{\delta V} + \frac{\delta\Delta\Gamma}{\delta v} - \frac{\delta}{\delta V}\langle\bar\eta, (i\mathbf{D})^{-1}\eta\rangle\\ &= -\frac{1}{2G_V}V + \Big(\frac{\delta}{\delta v} - \frac{\delta}{\delta V}\Big)\Delta\Gamma\end{aligned} \tag{5.15}$$

where in the last step we have used the equations of motion (5.14). For definiteness we will refer to them as the self-consistent currents. Notice that through the equations of motion the dynamical fields acquire a dependence on the external bosonic and fermionic fields. One can prove that the implicit dependence on the external fermionic fields can be neglected in the limit $N_c \to \infty$ as well as the bosonic integration. Thus we get in this limit

$$\begin{aligned}&\langle 0|T\Big[J_V(x)\exp\Big\{i\int d^4x\mathcal{L}_{\mathrm{ext}}\Big\}\Big]|0\rangle \to\\ &\to \Big(-\frac{1}{2G_V}V + \Big[\frac{\delta}{\delta v} - \frac{\delta}{\delta V}\Big]\Delta\Gamma\Big)\langle 0|T\exp\Big\{i\int d^4x\mathcal{L}_{\mathrm{ext}}\Big\}\Big]|0\rangle\end{aligned} \tag{5.16}$$

If we set the external bosonic fields equal to zero then we get that the total currents are given by

$$
\begin{aligned}
J_\mu^V =& -\frac{1}{2G_V}V_\mu + \\
& -\frac{iN_c}{24\pi^2}\epsilon_{\mu\nu\alpha\beta}\Big[i\{\partial^\nu A^\alpha, V^\beta\} + i\{\partial^\nu V^\alpha, A^\beta\} + A^\nu A^\alpha A^\beta + \{V^\nu V^\alpha, A^\beta\} + V^\nu A^\alpha V^\beta\Big] \\
J_\mu^A =& -\frac{1}{2G_V}A_\mu + \\
& -\frac{iN_c}{24\pi^2}\epsilon_{\mu\nu\alpha\beta}\Big[i\{\partial^\nu V^\alpha, V^\beta\} + i\{\partial^\nu A^\alpha, A^\beta\} + \{V^\nu, A^\alpha A^\beta\} + V^\nu V^\alpha V^\beta + A^\nu V^\alpha A^\beta\Big]
\end{aligned}
\tag{5.17}
$$

These equations represent the leading N_c modifications to the usual Current-Field Identities [22] and as we see they are valid in the presence of external quark fields (see eq. (5.16)). In particular, they can be used to evaluate baryon matrix elements or form factors. It is interesting to notice that in terms of right and left field representation (see Appendix A) these equations can be rewritten as

$$
\begin{aligned}
J_\mu^R &= -\frac{1}{2G_V}V_\mu^R + \frac{N_c}{24\pi^2}\epsilon_{\mu\nu\alpha\beta}\Big[\frac{1}{2}\{V_R^{\nu\alpha}, V_R^\beta\} + iV_R^\nu V_R^\alpha V_R^\beta\Big]; \\
J_\mu^L &= -\frac{1}{2G_V}V_\mu^L + \frac{N_c}{24\pi^2}\epsilon_{\mu\nu\alpha\beta}\Big[\frac{1}{2}\{V_L^{\nu\alpha}, V_L^\beta\} + iV_L^\nu V_L^\alpha V_L^\beta\Big];
\end{aligned}
\tag{5.18}
$$

where

$$
V_R^{\mu\nu} = \partial^\mu V_R^\nu - \partial^\mu V_R^\nu - i[V_R^\mu, V_R^\nu]; \qquad V_L^{\mu\nu} = \partial^\mu V_L^\nu - \partial^\mu V_L^\nu - i[V_L^\mu, V_L^\nu] \tag{5.19}
$$

Let us remind that in writing the former expressions explicit use of the equations of motion has been made. It is interesting to note that the correction to the current field identities coincide with the difference between the covariant and consistent currents related to the covariant and consistent forms of the anomaly respectively [33]. Thus, these currents do not posses an internal anomaly.

6. Counterterms from the Gauged Wess-Zumino action

In this section we describe the constructive method that we have used to compute the counterterms $\Delta\Gamma$ given by eq. (5.7). We remind that they have to be summed to the additive vector gauge invariant fermionic action $-i\log\overline{\mathrm{Det}}(i\mathbf{D})$ so that the total sum reproduces the QCD anomaly. To do so we will proceed by considering the gauged Wess-Zumino term. This represents no limitation since the anomaly is saturated by it. In addition, it should be kept in mind that the final expression for $\Delta\Gamma$ has to be a polynomial in the fields.

The left-right gauged Wess-Zumino term as a functional of the additive field combinations $\mathcal{A}_R$ and $\mathcal{A}_L$ reads (see e.g. [34] for an explicit construction)

$$
\begin{aligned}
\Gamma_{\mathrm{WZ}}^{\mathrm{RL}}[U, \mathcal{A}_R, \mathcal{A}_L] = \Gamma_{\mathrm{WZ}}[U] & \\
& + \frac{N_c}{48\pi^2}\int \mathrm{tr}\,\Big[\mathcal{A}_R U_R^3 + i\{\mathcal{A}_R, d\mathcal{A}_R\}U_R + iU^\dagger \mathcal{A}_L U \mathcal{A}_R U_R^2 \\
& + iU^\dagger \mathcal{A}_L U d\mathcal{A}_R U_R + \frac{i}{2}(\mathcal{A}_R U_R)^2 + \mathcal{A}_R^3 U_R - U^\dagger \mathcal{A}_L U\{\mathcal{A}_R, d\mathcal{A}_R\} \\
& - U^\dagger \mathcal{A}_L U \mathcal{A}_R U_R \mathcal{A}_R + iU^\dagger \mathcal{A}_L U \mathcal{A}_R^3 + \frac{i}{4}(U^\dagger \mathcal{A}_L U \mathcal{A}_R)^2\Big] - \text{p.c.}
\end{aligned}
\tag{6.1}
$$

where p.c. means interchanging the R and L labels and $\Gamma_{\mathrm{WZ}}[U]$ represents the topological Wess-Zumino action [4] (see Appendix D) and U a unitary flavour field $U^\dagger U = 1$. Here the following 1-forms have been defined

$$
\mathcal{A}_R = v_R + V_R; \qquad \mathcal{A}_L = v_L + V_L; \qquad U_R = U^\dagger dU; \qquad U_L = U dU^\dagger \tag{6.2}
$$

Under a chiral transformation the fields transform as follows

$$
\begin{aligned}
&\delta v_R = d\epsilon_R + i[\epsilon_R, v_R]; \qquad &&\delta v_L = d\epsilon_L + i[\epsilon_L, v_L] \\
&\delta V_R = +i[\epsilon_R, V_R]; &&\delta V_L = +i[\epsilon_L, V_L] \\
&\delta U = i(\epsilon_L U - U\epsilon_R); &&\delta U^\dagger = i(\epsilon_R U^\dagger - U^\dagger \epsilon_R)
\end{aligned}
\tag{6.3}
$$

i.e., the dynamical fields transform homogeneously, whereas the external fields transform non-homogeneously. The variation gives the right-left form of the anomaly

$$
\delta\Gamma^{\mathrm{RL}}_{\mathrm{WZ}}[U, \mathcal{A}_R, \mathcal{A}_L] = \frac{N_c}{48\pi^2}\int \mathrm{tr}\left[\Big(\{\mathcal{A}_R, d\mathcal{A}_R\} - i\mathcal{A}_R^3\Big)d\epsilon_R\right] - \mathrm{p.c.} \tag{6.4}
$$

To bring this anomaly to a vector gauge invariant form we consider the polynomial action

$$
\Gamma^{\mathrm{RL}}_{\mathrm{WZ}}[1, \mathcal{A}_R, \mathcal{A}_L] = \frac{N_c}{48\pi^2}\int \mathrm{tr}\left[-\mathcal{A}_L\{\mathcal{A}_R, d\mathcal{A}_R\} + i\mathcal{A}_L\mathcal{A}_R^3 + \frac{i}{4}(\mathcal{A}_L\mathcal{A}_R)^2\right] - \mathrm{p.c.} \tag{6.5}
$$

where 1 means the unit matrix in flavour space ($U = 1$ in (6.1)). The anomaly (5.4) can be reproduced by the effective action

$$
\Gamma^{V}_{\mathrm{WZ}}[U, \mathcal{A}_R, \mathcal{A}_L] = \Gamma^{\mathrm{RL}}_{\mathrm{WZ}}[U, \mathcal{A}_R, \mathcal{A}_L] - \Gamma^{\mathrm{RL}}_{\mathrm{WZ}}[1, \mathcal{A}_R, \mathcal{A}_L]; \qquad \delta\Gamma^{V}_{\mathrm{WZ}}[U, \mathcal{A}_R, \mathcal{A}_L] = G_A[\mathcal{V}, \mathcal{A}] \tag{6.6}
$$

This action coincides with that of ref. [23] and any other action reproducing the anomaly in terms of additive fields (5.4) differs by chirally invariant terms from this one. Actually, the former action (6.6) is the leading contribution in a gradient expansion of the γ_5-odd part of $\log \overline{\mathrm{Det}}(i\mathbf{D})$ provided the non-linearly transforming field U is taken as the (unique) unitary part of the flavour matrices $S + iP$. Hence, the terms of action (6.6) containing the fields either U or $U^\dagger$ are not polynomial actions due to the chiral circle condition $U^\dagger U = 1$. This anomaly, however, <u>does not</u> coincide with the QCD anomaly [1], since it contains the dynamical vector and axial fields V and A. As we know the QCD anomaly depends on the external fields v and a only. To eliminate the internal fields dependence we propose the most general globally chiral invariant counterterm depending on vector and axial degrees of freedom in a way that their variation exactly cancels the dependence on the internal fields in the anomaly.

$$
\begin{aligned}
\Gamma^{\mathrm{RL}}_{\mathrm{ct}}[v_R, v_L; V_R, V_L] = \int \mathrm{tr}\Big\{&c_1 V_R v_R^3 + c_2 V_R dv_R v_R + c_3 V_R v_R dv_R + c_4 V_R^2 v_R^2 \\
&+ c_5 (V_R v_R)^2 + c_6 V_R^2 dv_R + c_7 dV_R V_R v_R + c_8 V_R^3 v_R\Big\} - \mathrm{p.c.}
\end{aligned}
\tag{6.7}
$$

Under a chiral transformation we have

$$
\begin{aligned}
\delta\Gamma^{\mathrm{RL}}_{\mathrm{ct}} = \int \mathrm{tr}\Big\{\Big[&(c_1 + ic_2)v_R^2 V_R + (c_1 + ic_3)V_R v_R^2 - (c_1 + ic_2 + ic_3)v_R V_R v_R \\
&+ c_2 V_R dv_R + c_3 dv_R V_R - (c_4 + ic_6)v_R V_R^2 + (c_4 + ic_6 - ic_7)V_R^2 v_R \\
&+ (2c_5 + ic_7)V_R v_R V_R + c_7 dV_R V_R + c_8 V_R^3\Big]d\epsilon_R\Big\} - \mathrm{p.c.}
\end{aligned}
\tag{6.8}
$$

The coefficients $c_1, \dots, c_8$ are to be fixed by imposing that

$$
\delta\Big(\Gamma^{\mathrm{RL}}_{\mathrm{WZ}}[U, \mathcal{A}_R, A_L] - \Gamma^{\mathrm{RL}}_{\mathrm{ct}}[v_R, v_L; V_R, V_L]\Big) = \frac{N_c}{48\pi^2}\int \mathrm{tr}\left[\Big(\{v_R, dv_R\} - iv_R^3\Big)d\epsilon_R\right] - \mathrm{p.c.} \tag{6.9}
$$

This equation fixes all coefficients except one

$$
\begin{aligned}
\Gamma^{\mathrm{RL}}_{\mathrm{ct}}[v_R, v_L; V_R, V_L] = -\frac{iN_c}{48\pi^2}\int \mathrm{tr}\Big[&3V_R v_R^3 + 2iV_R\{dv_R, v_R\} + c\, V_R^2(dv_R - iv_R^2) \\
&+ i\{dV_R, V_R\}v_R + \frac{3}{2}(V_R v_R)^2 + V_R^3 v_R\Big] - \mathrm{p.c.}
\end{aligned}
\tag{6.10}
$$

The undetermined coefficient c vanishes if, in addition, we impose CP invariance. Indeed, under CP we have

$$\mathrm{CP}:\ (V_R)_\mu(t,\vec{x}) \to -(V_L)^{\mu,t}(t,-\vec{x}); \qquad (V_L)_\mu(t,\vec{x}) \to -(V_R)^{\mu,t}(t,-\vec{x}) \tag{6.11}$$

where upperscript t means matrix-transposed. After some integrations by parts one can see that CP results in changing $c \to -c$. We will take $c = 0$ for the rest of this section. A brief discussion of CP violating terms can be found in Appendix E.

The following action reproduces the correct anomaly (5.11)

$$\Gamma^V_{\rm WZ}[U,v,a;V,A] = \Gamma^{\rm RL}_{\rm WZ}[U,\mathcal{A}_R,\mathcal{A}_L] - \Gamma^{\rm RL}_{\rm ct}[v_R,v_L;V_R,V_L] - \Gamma^{\rm RL}_{\rm WZ}[1,v_R,v_L] \tag{6.12}$$

i.e. $\delta\Gamma^V_{\rm WZ}[U,v,a;V,A] = G_A[v,a]$. However, in a vector gauge invariant calculation of the fermion determinant the chirally breaking terms, which depend on the additive combinations $\mathcal{A}_R = v_R + V_R$ and $\mathcal{A}_L = v_L + V_L$, are given by eq. (6.6). Thus we define

$$\Delta\Gamma = \Gamma^{\rm RL}_{\rm WZ}[1,\mathcal{A}_R,\mathcal{A}_L] - \Gamma^{\rm RL}_{\rm WZ}[1,v_R,v_L] - \Gamma^{\rm RL}_{\rm ct}[v_R,v_L;V_R,V_L] \tag{6.13}$$

so that

$$\Gamma^V_{\rm WZ}[U,v,a;V,A] = \Gamma^V_{\rm WZ}[U,\mathcal{A}_R,\mathcal{A}_L] + \Delta\Gamma[v,a;V,A] \tag{6.14}$$

Thus, $\Delta\Gamma[v,a;V,A]$ are the counterterms to be added to the vector gauge invariant fermion determinant $\overline{\rm Det}(i\mathbf{D})$, determined up to local and polynomial chiral gauge invariant combinations. These combinations must be formed by using the chirally covariant objects F_R, DV_R and V_R in all possible combinations. For completeness we repeat here the argument already given in ref. [32]. If we insist on parity conservation the following chiral gauge invariant combinations remain

$$\mathrm{tr}\, DV_R V_R^2, \quad \mathrm{tr}\, F_R V_R^2, \quad \mathrm{tr}\, DV_R F_R, \quad \mathrm{tr}\,(DV_R)^2, \quad \mathrm{tr}\, V_R^4, \quad \mathrm{tr}\, F_R^2 \tag{6.15}$$

together with the corresponding left-field combinations. The first term vanishes identically as can be seen integrating by parts. The second term is the CP violating term already considered before. The third term vanishes after integration by parts due to the Bianchi identity for the field strength tensor $DF_R = 0$. The fourth term can be reduced integrating by parts to the second one. The fifth also vanishes identically due to the cyclic property of the trace. Finally, the sixth term is a topological action in the external fields which are assumed to have zero winding and hence vanishes. This completes the proof that the counterterms are uniquely given if CP invariance is invoked.

7. Study of the two flavour case

It is interesting to study the two flavour reduction of the model which corresponds to a NJL Lagrangian of the form

$$\begin{aligned}\mathcal{L} = \bar{q}(i\partial\!\!\!/ - \hat{M}_0)q + \frac{G_1}{2}\Big[(\bar{q}q)^2 + (\bar{q}\vec{\tau}i\gamma_5 q)^2\Big] \\ - \frac{G_2}{2}\Big[(\bar{q}\vec{\tau}\gamma_\mu q)^2 + (\bar{q}\vec{\tau}\gamma_\mu\gamma_5 q)^2\Big] - \frac{G_3}{2}(\bar{q}\gamma_\mu q)^2\end{aligned} \tag{7.1}$$

Notice that, strictly speaking, the case $G_2 \neq G_3$ is not a particular case of the model considered in section 2, however trivial modifications can be easily implemented to the case of interest. After bosonization we get

$$\begin{aligned}\mathcal{L} = \bar{q}\Big(i\partial\!\!\!/ - g_\pi(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi}) + \frac{g_\rho}{2}\vec{\tau}\cdot(\vec{\rho\!\!\!/} + \vec{A\!\!\!/}\gamma_5) + g_\omega\omega\!\!\!/ - \hat{M}_0\Big)q \\ - \frac{1}{2}\mu^2(\sigma^2 + \vec{\pi}^2) + \frac{1}{2}m_\rho^2(\vec{\rho}^\mu\cdot\vec{\rho}_\mu + \vec{A}^\mu\cdot\vec{A}_\mu) + \frac{1}{2}m_\omega^2\omega_\mu\omega^\mu\end{aligned} \tag{7.2}$$

with $G_1 = g_\pi^2/\mu^2$, $G_2 = g_\rho^2/(4m_\rho^2)$ and $G_3 = g_\omega^2/m_\omega^2$. Up to the mass term, this Lagrangian is invariant under the $\mathrm{SU}(2)_R \otimes \mathrm{SU}(2)_L \otimes \mathrm{U}_B(1)$ group and has been studied in detail in refs. [29,30,31]. The corresponding baryon, vector and axial currents are

$$J^B_\mu(x) = \bar{q}(x)\gamma_\mu q(x); \qquad \vec{J}^V_\mu(x) = \frac{1}{2}\bar{q}(x)\gamma_\mu\vec{\tau}q(x); \qquad \vec{J}^A_\mu(x) = \frac{1}{2}\bar{q}(x)\gamma_\mu\gamma_5\vec{\tau}q(x) \tag{7.3}$$

In the notation of previous sections the reduction of the Lagrangian corresponds to take the dynamical fields to be

$$V_\mu = \frac{1}{2} g_\rho \vec{\rho}_\mu \cdot \vec{\tau} + g_\omega \omega_\mu ; \qquad A_\mu = \frac{1}{2} g_\rho \vec{A}_\mu \cdot \vec{\tau} \tag{7.4}$$

whereas the external fields reduce to

$$v_\mu = \frac{1}{2} \vec{v}_\mu \cdot \vec{\tau} + v^0_\mu ; \qquad a_\mu = \frac{1}{2} \vec{a}_\mu \cdot \vec{\tau} \tag{7.5}$$

In this model the Ward identities acquire a simple form in the case of vanishing s, p, η and $\bar{\eta}$ external fields,

$$\begin{aligned}
\partial^\mu J^B_\mu &= 0 \\
\partial^\mu \vec{J}^V_\mu &= \vec{J}^V_\mu \wedge \vec{v}^\mu + \vec{J}^A_\mu \wedge \vec{a}^\mu \\
\partial^\mu \vec{J}^A_\mu &= \vec{J}^A_\mu \wedge \vec{v}^\mu + \vec{J}^V_\mu \wedge \vec{a}^\mu - \frac{N_c}{4\pi^2} \epsilon^{\mu\nu\alpha\beta} \partial_\mu v^0_\nu \Big(\partial_\alpha \vec{v}_\beta + \frac{1}{2} \vec{v}_\alpha \wedge \vec{v}_\beta - \frac{1}{2} \vec{a}_\alpha \wedge \vec{a}_\beta \Big)
\end{aligned} \tag{7.6}$$

The modification of the vector gauge invariant regularized effective action can then be obtained directly from eqs. (5.12) giving

$$\begin{aligned}
\Delta\Gamma_0 = \frac{N_c}{12\pi^2} g^2_\rho g_\omega \int d^4x \epsilon_{\mu\nu\alpha\beta} \Big\{ & \partial^\mu \omega^\nu \vec{A}^\alpha \cdot \vec{\rho}^\beta + \partial^\mu \vec{\rho}^\nu \cdot \vec{A}^\alpha \omega^\beta \\
& - \frac{g_\rho}{8} \omega^\mu [\vec{A}^\nu \wedge \vec{A}^\alpha + 3 \vec{\rho}^\nu \wedge \vec{\rho}^\alpha] \cdot \vec{A}^\beta \Big\}
\end{aligned} \tag{7.7}$$

In this particular case the total baryon, isospin and axial currents, in the absence of external fields, become

$$J^\mu_B = \frac{\delta W}{\delta v^0_\mu}\Big|_0 \qquad \vec{J}^\mu_V = \frac{\delta W}{\delta \vec{v}_\mu}\Big|_0 \qquad \vec{J}^\mu_A = \frac{\delta W}{\delta \vec{a}_\mu}\Big|_0 \tag{7.8}$$

respectively. Straightforward calculation yields the following results for the modification of the currents as obtained from eq. (5.12)

$$\begin{aligned}
\Delta J^B_\mu &= \frac{N_c}{8\pi^2} g^2_\rho \epsilon_{\mu\nu\alpha\beta} \partial^\nu (\vec{A}^\alpha \cdot \vec{\rho}^\beta) \\
\Delta \vec{J}^V_\mu &= \frac{N_c}{24\pi^2} g_\rho g_\omega \epsilon_{\mu\nu\alpha\beta} \Big[3 \partial^\nu (\vec{A}^\alpha \omega^\beta) + 2 g_\rho \omega^\nu (\vec{A}^\alpha \wedge \vec{\rho}^\beta) \Big] \\
\Delta \vec{J}^A_\mu &= \frac{N_c}{24\pi^2} g_\rho g_\omega \epsilon_{\mu\nu\alpha\beta} \Big[3 \vec{\rho}^\nu \partial^\alpha \omega^\beta + 3 \partial^\nu \vec{\rho}^\alpha \omega^\beta + g_\rho \omega^\nu (\vec{A}^\alpha \wedge \vec{A}^\beta + \vec{\rho}^\alpha \wedge \vec{\rho}^\beta) \Big]
\end{aligned} \tag{7.9}$$

Notice that the correction to the baryon current is a total divergence, hence the baryon number normalization is preserved. The total self-consistent currents, i.e. the total currents evaluated by means of the equations of motion can be deduced from eqs. (5.17) and give

Baryon Current

$$J^B_\mu = -\frac{m^2_\omega}{g_\omega} \omega_\mu + \frac{N_c}{24\pi^2} g^2_\rho \epsilon_{\mu\nu\alpha\beta} \Big\{ \partial^\nu \vec{A}^\alpha \cdot \vec{\rho}^\beta + \partial^\nu \vec{\rho}^\alpha \cdot \vec{A}^\beta + \frac{1}{4} g_\rho (\vec{A}^\nu \wedge \vec{A}^\alpha + 3 \vec{\rho}^\nu \wedge \vec{\rho}^\alpha) \cdot \vec{A}^\beta \Big\} \tag{7.10}$$

Vector Current

$$\vec{J}^V_\mu = -\frac{m^2_\rho}{g_\rho} \vec{\rho}_\mu + \frac{N_c}{24\pi^2} g_\rho g_\omega \epsilon_{\mu\nu\alpha\beta} \Big\{ \partial^\nu \vec{A}^\alpha \omega^\beta + \partial^\nu \omega^\alpha \vec{A}^\beta + \frac{1}{2} g_\rho \omega^\nu (\vec{\rho}^\alpha \wedge \vec{A}^\beta) \Big\} \tag{7.11}$$

Axial Current

$$\vec{J}^A_\mu = -\frac{m^2_\rho}{g_\rho} \vec{A}_\mu + \frac{N_c}{24\pi^2} g_\rho g_\omega \epsilon_{\mu\nu\alpha\beta} \Big\{ \partial^\nu \omega^\alpha \vec{\rho}^\beta + \partial^\nu \vec{\rho}^\alpha \omega^\beta + \frac{1}{4} g_\rho \omega^\nu (\vec{A}^\alpha \wedge \vec{A}^\beta + \vec{\rho}^\alpha \wedge \vec{\rho}^\beta) \Big\} \tag{7.12}$$

These are the corrected current-field identities in the two flavour case in leading order in N_c. To conclude this section we notice that if the fields $\vec{\rho}_\mu$ and $\vec{A}_\mu$ vanish, there are no corrections to any single current. In contrast, if the field ω_μ vanishes there is a correction to the baryon current not considered in previous works [26,27].

8. Numerical Results for Nucleon Observables

Following the standard approach [11], a baryon can be described in terms of the corresponding correlation function

$$\Pi_B(x,x') = \langle 0|T\Big\{B(x)\bar{B}(x')\Big\}|0\rangle \tag{8.1}$$

$B(x)$ being a baryonic operator in terms of quark fields. We take

$$B(x) = \frac{1}{N_c!}\epsilon^{\alpha_1,\dots,\alpha_{N_c}}\Phi^{a_1,\dots,a_{N_c}} q_{\alpha_1 a_1}(x)\cdots q_{\alpha_{N_c} a_{N_c}}(x) \tag{8.2}$$

where $(\alpha_1,\dots,\alpha_{N_c})$ are colour indices, $(a_1,\dots,a_{N_c})$ spinor-flavour indices and $\Phi^{a_1,\dots,a_{N_c}}$ the proper completely symmetric spinor-flavour amplitude. The exact spectral representation of the correlation function is obtained as usual by inserting the complete set of eigenstates of the NJL hamiltonian in eq. (8.1), namely,

$$\begin{aligned}\Pi_B(x,x') = {}& \theta(t-t')\sum_n \langle 0|B(0)|B_n,\vec{k}\rangle\langle B_n,\vec{k}|\bar{B}(0)|0\rangle e^{-i(x-x')k} \\ & + (-1)^{N_c}\theta(t'-t)\sum_n \langle 0|\bar{B}(0)|\bar{B}_n,\vec{k}\rangle\langle \bar{B}_n,\vec{k}|B(0)|0\rangle e^{+i(x-x')k}\end{aligned} \tag{8.3}$$

where B_n, $(\bar{B}_n)$ are the baryonic (antibaryonic) states with momentum $\vec{k}$. Further, by chosing the branch $t > t'$ and taking the limit $t - t' = T \to -i\infty$, the lightest baryon, and at rest, is selected in the sum,

$$\Pi_B(x,x') = \langle 0|B(0)|B\rangle\langle B|\bar{B}(0)|0\rangle e^{-iTM_B} \tag{8.4}$$

To carry out these steps in the large N_c limit, we first write the time ordered product (8.1) as a path integral over fermionic degrees of freedom with weight $\exp(iS_{\rm NJL})$. The resulting expression, in turn, can be obtained by appropriate functional differentiation of the generating functional $Z[s,p,v,a,\eta,\bar{\eta}]$ (see eq.(2.3)) with respect to the external quark fields $\eta(x)$ and $\bar{\eta}(x)$. After bosonization and integration of the quarks one gets, using eq.(2.7),

$$\Pi_B(x,x') = \Phi^{a_1,\dots,a_{N_c}}\bar{\Phi}^{a'_1,\dots,a'_{N_c}} \frac{\int DM_{\rm int}\exp(i\overline{W})\prod_{i=1}^{N_c} iS_{a_i a'_i}(x,x')}{\int DM_{\rm int}\exp(i\overline{W})} \tag{8.5}$$

where the one particle Green function $S_{aa'}(x,x') = \langle x|(i\mathbf{D}+i\epsilon)^{-1}_{aa'}|x'\rangle$ has been introduced and all external fields s,p,v,a and $\bar{\eta},\eta$ are set to zero. The limit $N_c\to\infty$ drives the functional integral in the denominator to a saddle point bosonic configuration which describes the mean field vacuum. On the other hand, because there are N_c factors $S_{a,a'}$ in the numerator, the dominating saddle point configuration will be different from that of the vacuum and will depend on x and x'. Next, the limit of large evolution time T selects the minimum energy stationary configuration. For stationary configurations one can use the spectral representation of the propagator given by

$$iS_{aa'}(\vec{x},t;\vec{x}',t') = \sum_n (\theta(t-t')\theta(\epsilon_n) - \theta(t'-t)\theta(-\epsilon_n))\psi_{na}(\vec{x})\bar{\psi}_{na'}(\vec{x}')e^{-i\epsilon_n(t-t')} \tag{8.6}$$

with $\psi_{na}(\vec{x})$ and ϵ_n the eigenfunctions and eigenvalues of the single particle Dirac Hamiltonian H defined by $i\mathbf{D} = \gamma_0(i\partial_t - H)$ (evaluated at the stationary bosonic configuration). Choosing the $t > t'$ branch to create a baryon (instead of an antibaryon) and taking the limit $t - t' = T \to -i\infty$ we get

$$\Pi_B(x,x') = \Psi_B(\vec{x})\bar{\Psi}_B(\vec{x}')e^{-iTM_B} \tag{8.7}$$

where the baryon wave function

$$\Psi_B(\vec{x}) = \Phi^{a_1,\ldots,a_{N_c}} \psi^{\rm val}_{a_1}(\vec{x}) \cdots \psi^{\rm val}_{a_{N_c}}(\vec{x}) \tag{8.8}$$

has been introduced. Here, the valence level $\epsilon_{\rm val}$ is the lowest positive eigenvalue of the Dirac Hamiltonian. This level is selected in eq. (8.6) by the large Euclidean time limit. Eq. (8.7) shows that spatial translational invariance is spontaneously broken by the mean field approximation, corresponding to the formation of a solitonic configuration in the presence of valence quarks.

The baryon mass (in the large N_c limit) comes from the two contributions in the numerator and denominator, $M_B = E_B - E_{\rm vac}$, which are given by

$$\begin{aligned} E_B &= \overline{E}^{\rm sol}_{\rm sea} + \Delta E^{\rm sol} + E^{\rm sol}_m + N_c \epsilon^{\rm val} \\ E_{\rm vac} &= \overline{E}^{\rm vac}_{\rm sea} + \Delta E^{\rm vac} + E^{\rm vac}_m \end{aligned} \tag{8.9}$$

where the superscripts "vac" and "sol" stands for the two different stationary saddle point configurations. The valence term (last term) in E_B, not present in $E_{\rm vac}$, comes from the propagators $S_{a,a'}$. The other terms come from the action in eq. (5.13). The sea regularized and the pure mesonic contribution are given by

$$\begin{aligned} \overline{E}_{\rm sea} + \Delta E &= \frac{i}{T} \log {\rm Det}(i\mathbf{D}) = \frac{i}{T} \Big[\log \overline{\rm Det}(i\mathbf{D}) + i\Delta\Gamma_0 \Big] \\ E_m &= -\frac{1}{T} W_m \end{aligned} \tag{8.10}$$

respectively. At this point it is clear that any expression for the total energy involving the Dirac eigenvalues will necessarily depend on the additive combinations. Similarly to the energy, one body observables, like e.g. mean squared radius, axial coupling constant, etc. admit (at leading order in N_c) a natural decomposition in terms of a valence part, a pure additively regularized part and the modifications induced by the counterterms. This fact can be established by repeating similar steps as for the baryon mass but with non vanishing external bosonic fields s, p, v and a. As an example we give such a decomposition for the baryon axial coupling constant

$$g^N_A = g^{\rm val}_A + \overline{g}^{\rm sea}_A + \Delta g^N_A \tag{8.11}$$

i.e. in the absence of counterterms, Δg^N_A would vanish.

For stationary fields the Dirac operator can be written as

$$i\mathbf{D} = \gamma_0 (i \frac{\partial}{\partial t} - H) \tag{8.12}$$

with the one particle Dirac Hamiltonian

$$H = -i\alpha \cdot \nabla - (V_0 + A_0\gamma_5) + \alpha \cdot (V + A\gamma_5) + \beta(S + i\gamma_5 P) \tag{8.13}$$

In the case of two flavours with hedgehog symmetry we have ($\hat{x} = \vec{x}/r$)

$$\begin{aligned} & S = g_\pi \sigma(r); \qquad P = g_\pi \tau \cdot \hat{x} \phi(r); \qquad V_0 = g_\omega \omega(r); \qquad A_0 = 0 \\ & \alpha \cdot V = \frac{g_\rho}{2} \alpha \cdot (\hat{x} \wedge \tau) \rho(r) \\ & \alpha \cdot A = \frac{g_\rho}{2} \Big[\alpha \cdot \tau \Big(A_S(r) - \frac{1}{3} A_T(r) \Big) + (\alpha \cdot \hat{x})(\tau \cdot \hat{x}) A_T(r) \Big] \end{aligned} \tag{8.14}$$

If we introduce the hedgehog ansatz in the expressions for energy (7.7) and the currents (7.9) we readily find the modifications for the total energy and the baryon, vector and axial currents. Their detailed form can be seen at Appendix C.

Given a vector gauge invariant regularization of the fermion determinant one should proceed as follows. The total energy is obtained by adding the valence quark, the Dirac sea contribution (vector additively regularized), the bosonizing terms and the counterterms given by formula (8.9). One should then minimize this total energy with respect to arbitrary variations in the fields $\sigma, \phi, \rho, A_S, A_T$ and ω, and determine the solution of the resulting equations of motion by some iterative method until convergence is achieved. In the present case, such a procedure requires many iterations [29, 30, 31]. In addition, there is presently some concern [39,40,41] about the validity of any of the schemes proposed so far [25,28,29,30,31]. It is clear, however, that in either case the proper anomalous structure has not been included since a vector additive regularization has always been considered, i.e. the counterterms ΔE are missing. Therefore, before embarking in a rather complicated calculation we make an order of magnitude estimate treating the counterterms as a small perturbation of the total energy. Clearly, such a procedure can be justified a posteriori if the modifications actually turn out to be small.

A quantitative estimate of the corrections can be obtained by considering any of the self-consistent solutions available in the literature [29,30,31], since in practice they do not differ too much from each other. On simple dimensional grounds and using the typical sizes of the mesonic fields ($\sim f_\pi$) and their typical extension (~ 1 fm) for reasonable values of the parameters we estimate the counterterm to the total energy to be of the order of few MeV. A more systematic estimate using the self-consistent solutions of ref. [29] gives values for $\Delta E < 10$ MeV. This small number does not stem neither from a big cancellation among the different terms in the expression for ΔE nor from cancellations between the interior and the exterior of the soliton. The size of the correction is to be compared to the typical soliton energies $E \sim 1500$ MeV. Thus it seems more than reasonable to treat the counterterms perturbatively. For the currents a similar strategy can be considered. In this case we can compare the perturbative contributions to the isoscalar nucleon radius and the axial coupling constant to the self-consistent ones given by formulas (C.5), (C.6), (C.7) and (C.10) respectively, the difference being an indication of how far are we from a self-consistent solution, i.e. they represent virial theorems. Again the order of magnitude estimate predicts $\Delta r \sim 0.01$ fm and $\Delta g_A^N \sim 0.01$ in any calculation scheme, much smaller than the typical values usually found ($r \sim 0.8$ fm and $g_A^N \sim 0.5$ [29,30,31]). This result is also confirmed by an accurate calculation using the solutions of ref. [29]. It is beyond doubt that the modification of the Nambu–Jona-Lasinio model to incorporate the proper QCD anomaly does not result in large changes for the computed nucleon properties. At first sight this is a bit surprising since in the meson sector 20% deviations with respect to the current algebra result have been obtained for $\gamma \to \pi\pi\pi$ [23]. On the other hand, the corrections induced by the counterterms involve vector mesons only (see Appendix C) whose typical extension is $1/m_\rho$. Due to this the corrections in the nucleon currents are of order $f_\pi/m_\rho \sim 0.1$, i.e. they are small because of the large vector meson masses. It is interesting to note here that similar trends have been also found in the meson sector when calculating full momentum dependent abnormal parity vertex functions [42].

9. Effective action for Vector Mesons up to fourth order in momenta

In this section we further exploit our results to write down an effective action for vector mesons and external fields up to fourth order in momenta. The interesting point is that the anomaly does not imply the coupling strength of strong decay processes like $\omega \to 3\pi$, $\omega \to \rho\pi$ or radiative vector meson decays like $\rho \to \gamma\pi$ etc., although the corresponding amplitudes have abnormal pseudoparity (i.e. contain an $\epsilon_{\mu\nu\alpha\beta}$ tensor).

As we have said in section 6, up to fourth order the correct anomalous action reproducing the proper QCD anomaly is given by

$$\Gamma^V_{\mathrm{WZ}}[U, v, a; V, A] = \Gamma^{\mathrm{RL}}_{\mathrm{WZ}}[U, \mathcal{A}_R, \mathcal{A}_L] - \Gamma^{\mathrm{RL}}_{\mathrm{ct}}[v, V] - \Gamma^{\mathrm{RL}}_{\mathrm{WZ}}[1, v_R, v_L] \tag{9.1}$$

On the other hand the action

$$\Gamma^V_{\mathrm{WZ}}[U, v, a] = \Gamma^{\mathrm{RL}}_{\mathrm{WZ}}[U, v_R, v_L] - \Gamma^{\mathrm{RL}}_{\mathrm{WZ}}[1, v_R, v_L] \tag{9.2}$$

corresponding to $G_V = 0$ and $\Delta\Gamma = 0$ also reproduces the QCD anomaly. Therefore both actions differ by chirally covariant terms. A direct computation gives

$$
\begin{aligned}
\Gamma^V_{\mathrm{WZ}}[U,v,a;V,A] = \Gamma^V_{\mathrm{WZ}}[U,v,a] & \\
- \frac{iN_c}{48\pi^2}\int \mathrm{tr}\Big\{ & 2R\{F_R,V_R\} + DV_R[R,V_R] - iR(R^2+V_R^2)V_R \\
& + \frac{1}{2}(RV_R)^2 + 2iUF_RV_RU^\dagger V_L + URF_RU^\dagger V_L + iUV_RU^\dagger V_LF_L \\
& + UF_RRU^\dagger V_L + DV_RRU^\dagger V_LU + iDV_R[V_R,U^\dagger V_LU] + UR^2V_RU^\dagger V_L \\
& + iUV_RU^\dagger V_LLV_L + UV_RU^\dagger V_L^3 + \frac{1}{4}(UV_RU^\dagger V_L)^2\Big\} - \text{p.c.}
\end{aligned}
\tag{9.3}
$$

where the following chirally covariant 1-forms

$$
\begin{aligned}
R &= U^\dagger\nabla U = U^\dagger dU - iU^\dagger v^L U + iv^R \\
L &= U\nabla U^\dagger = UdU^\dagger - iUv^RU^\dagger + iv^L
\end{aligned}
\tag{9.4}
$$

and 2-forms

$$
\begin{aligned}
&F_R = dv_R - iv_R^2; \quad F_L = dv_L - iv_L^2 \\
&DV_R = dV_R - i\{V_R,v_R\}; \quad DV_L = dV_L - i\{V_L,v_L\}
\end{aligned}
\tag{9.5}
$$

have been introduced. The transformation properties of these objects are

$$
\delta R = i[\epsilon_R, R]; \qquad \delta F_R = i[\epsilon_R, F_R]; \qquad \delta DV_R = i[\epsilon_R, DV_R] \tag{9.6}
$$

similarly for the left combinations. It is important to mention that from the point of view of an effective mesonic theory each term appearing in eq. (9.3) can have an arbitrary coefficient since they are separately chirally invariant and do not contribute to the anomaly equation. The coefficients in eq. (9.3) represent the particular prediction of the Nambu–Jona-Lasinio model with vector mesons so defined to satisfy the correct QCD anomaly. Another important point is that these terms describe low energy vector meson strong and radiative abnormal parity decays although they are clearly not anomalous. In particular, if the external fields are set equal to zero the total action is invariant under global chiral transformations, and no internal anomaly appears. We will not attempt to study the phenomenological implications of the former action. Some of them have been considered in ref. [43] for low momenta and in ref. [42] in a full momentum dependent formalism, although the explicit form (9.3) has not been given. Finally, as a partial check of our results, we consider the low energy limit of the former abnormal parity non-anomalous action. This corresponds to integrate vector mesons out at the mean field level or equivalently in the large N_c limit. In the lowest relevant order we have the following equations of motion (see [35] and [29] for more details)

$$
V^R_\mu = \frac{i}{2}(1-g_A^Q)R_\mu; \qquad V^L_\mu = \frac{i}{2}(1-g_A^Q)L_\mu \tag{9.7}
$$

with g_A^Q the quark axial coupling constant [23,35]. Moreover

$$
\begin{aligned}
DV_R &= \frac{i}{2}(1-g_A^Q)\Big(-R^2 + iF_R - iU^\dagger F_LU\Big) \\
DV_L &= \frac{i}{2}(1-g_A^Q)\Big(-L^2 + iF_L - iUF_RU^\dagger\Big)
\end{aligned}
\tag{9.8}
$$

giving

$$
\Gamma^V_{\mathrm{WZ}}[U,v,a;V,A] = \Gamma^V_{\mathrm{WZ}}[U,v,a] + \cdots \tag{9.9}
$$

where the dots denote terms of order sixth at least. Therefore up to fourth order in the chiral expansion there are no corrections to the effective abnormal parity action. This result is also a consequence of the uniqueness of the counterterms to the Wess-Zumino action (see the discussion at the end of section 6).

10. Conclusions and Summary

In the present work we have investigated the anomalous sector of the Nambu–Jona-Lasinio model with vector mesons and its possible implications on the properties of the nucleon described as a system of three bound valence quarks in a self-consistent solitonic background of σ, π, ρ, A and ω mesons. In most cases, calculations within this model have ignored the fact that it does not reproduce the correct QCD chiral anomaly. For the model to do so for non-vanishing vector meson fields it is necessary to modify the usual definition of the fermionic determinant. This can be done by subtracting suitable local and polynomial counterterms to the action in the vector dynamical and external fields. This represents another solution which was overlooked in [23], hence the conclusion there, that the NJL model with vector mesons cannot reproduce the correct QCD anomaly, is incorrect. These counterterms only modify the abnormal parity vertices at zero momentum transfer and in the chiral limit, and hence leave many meson properties such as meson propagators and the momentum dependence of mesonic form factors unaffected. However, there appear abnormal parity modifications appear in the Current-Field identities at leading order in large N_c. In the soliton sector clear corrections appear in the nucleon mass and the axial and vector currents. For hedgehog profiles we have evaluated the numerical corrections to the soliton energy, the isoscalar nucleon radius and the axial coupling constant induced by the counterterms. We have found that they account for less than 1% of the total magnitude of the computed observables. As the counterterms only involve vector mesonic fields, and the corrections take place at zero momentum in the amplitudes, they are mainly sensitive to the tail of the vector meson fields only. The smallness of the counterterms in the nucleon might be understood due to the high vector meson masses. We conclude that the fact that the previous regularizations of the Nambu–Jona-Lasinio model do not fulfill the proper QCD chiral anomaly does not have practical dramatic consequences in the previous solitonic calculations.

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Appendix A. Chiral Transformations of the Dirac operator in Minkowski space

We define the effect of local chiral transformations on the Dirac operator, both in vector-axial notation as well as in right-left notation in Minkowski space. We follow Bjorken-Drell [44] (Itzykson-Zuber [45]) notation for the gamma matrices throughout.

A.1 Vector-Axial Notation

The Dirac operator $\mathbf{D}$ in Minkowski space is taken to be

$$i\mathbf{D} = i\not{\partial} + \not{\mathcal{V}} + \not{\mathcal{A}}\gamma_5 - (\mathcal{S} + i\gamma_5\mathcal{P}) \tag{A.1}$$

with S,P,V_μ,A_μ hermitean flavour fields $S = \frac{1}{2}\sum_{a=0}^{N_F^2-1} \lambda_a S^a(x), \cdots$, with the flavour matrices λ_a normalized as $\mathrm{tr}\,(\lambda_a\lambda_b) = 2\delta_{ab}$ and $\lambda_a\lambda_b = (d_{abc} + i\,f_{abc})\lambda_c$ with $a, b, c = 0, \ldots, N_F^2 - 1$. Under chiral (vector and axial) local transformations the Dirac operator transforms as

$$\mathbf{D} \to \mathbf{D}^g = e^{+i\epsilon_V(x)-i\epsilon_A(x)\gamma_5}\mathbf{D}e^{-i\epsilon_V(x)-i\epsilon_A(x)\gamma_5} \tag{A.2}$$

with

$$\epsilon_V(x) = \sum_a \epsilon_V^a(x)\lambda_a; \qquad \epsilon_A(x) = \sum_a \epsilon_A^a(x)\lambda_a \tag{A.3}$$

The induced transformations on the fields are given by

$$i\mathbf{D} \to i\mathbf{D}^g = i\not{\partial} + \not{\mathcal{V}}^g + \not{\mathcal{A}}^g\gamma_5 - (\mathcal{S}^g + i\gamma_5\mathcal{P}^g) \tag{A.4}$$

In the infinitesimal case this is equivalent to

$$\begin{aligned}
\delta\mathcal{V}^\mu &= [\mathcal{D}^\mu, \epsilon_V] + i[\epsilon_A, \mathcal{A}^\mu] \\
\delta\mathcal{A}^\mu &= i[\epsilon_V, \mathcal{A}^\mu] + [\mathcal{D}^\mu, \epsilon_A] \\
\delta\mathcal{S} &= i[\epsilon_V, \mathcal{S}] + \{\epsilon_A, \mathcal{P}\} \\
\delta\mathcal{P} &= i[\epsilon_V, \mathcal{P}] - \{\epsilon_A, \mathcal{S}\}
\end{aligned} \tag{A.5}$$

where the vector covariant derivative reads

$$\mathcal{D}_\mu = \partial_\mu - i\mathcal{V}_\mu \tag{A.6}$$

In addition, the dynamical and external quark fields satisfy the following transformation properties

$$\begin{aligned}
\delta q &= i(\epsilon_V + \epsilon_A\gamma_5)q \\
\delta\bar{q} &= -i\bar{q}(\epsilon_V - \epsilon_A\gamma_5) \\
\delta\eta &= i(\epsilon_V - \epsilon_A\gamma_5)\eta \\
\delta\bar{\eta} &= -i\bar{\eta}(\epsilon_V + \epsilon_A\gamma_5)
\end{aligned} \tag{A.7}$$

defined to make the bilinear forms $\bar{q}\mathbf{D}q$ and $\bar{\eta}\mathbf{D}^{-1}\eta$ invariant under chiral local transformations.

A.2 Left-Right Notation

We rewrite the Dirac operator as

$$\mathbf{D} = \mathbf{D}_R P_R + \mathbf{D}_L P_L \tag{A.8}$$

with the projection operators on chirality

$$P_R = \frac{1}{2}(1 + \gamma_5); \qquad P_L = \frac{1}{2}(1 - \gamma_5) \tag{A.9}$$

The right and left Dirac operators are given by

$$i\mathbf{D}_R = i\partial\!\!\!/ + \mathcal{A}\!\!\!/_R - \mathcal{M}; \qquad i\mathbf{D}_L = i\partial\!\!\!/ + \mathcal{A}\!\!\!/_L - \mathcal{M}^\dagger \tag{A.10}$$

with

$$\begin{aligned} &\mathcal{M} = \mathcal{S} + i\mathcal{P}; \qquad \mathcal{M}^\dagger = \mathcal{S} - i\mathcal{P}; \\ &\mathcal{A}^\mu_R = \mathcal{V}^\mu + \mathcal{A}^\mu; \qquad \mathcal{A}^\mu_L = \mathcal{V}^\mu - \mathcal{A}^\mu \end{aligned} \tag{A.11}$$

then we have the following transformation properties under infinitesimal chiral rotations

$$\begin{aligned} &\delta\mathcal{M} = i\epsilon_L\mathcal{M} - i\mathcal{M}\epsilon_R \\ &\delta\mathcal{M}^\dagger = i\epsilon_R\mathcal{M}^\dagger - i\mathcal{M}^\dagger\epsilon_L \\ &\delta\mathcal{A}^\mu_R = \partial^\mu\epsilon_R + i[\epsilon_R, \mathcal{A}^\mu_R] = [\mathcal{D}^\mu_R, \epsilon_R] \\ &\delta\mathcal{A}^\mu_L = \partial^\mu\epsilon_L + i[\epsilon_L, \mathcal{A}^\mu_L] = [\mathcal{D}^\mu_L, \epsilon_L] \end{aligned} \tag{A.12}$$

where

$$\epsilon_R = \epsilon_V + \epsilon_A; \qquad \epsilon_L = \epsilon_V - \epsilon_A; \qquad \mathcal{D}^R_\mu = \partial_\mu - i\mathcal{A}^R_\mu; \qquad \mathcal{D}^L_\mu = \partial_\mu - i\mathcal{A}^L_\mu \tag{A.13}$$

have been defined. Finally, the left and right currents are given by

$$J^{R,L}_\mu = J^V_\mu \pm J^A_\mu \tag{A.14}$$

Appendix B. Effective action, regularization and anomaly in Euclidean space

For completeness we give below our particular conventions for Euclidean space (hatted quantities)

$$\hat{x}^0 = ix^0; \qquad \hat{x}^i = x^i; \qquad \hat{x}^\mu = \hat{x}_\mu = (\hat{x}^0, \hat{x}^i) \tag{B.1}$$

$$\hat{V}^0(\hat{x}^0,\hat{x}^i) = iV^0(x^0,x^i); \qquad \hat{V}^i(\hat{x}^0,\hat{x}^i) = V^i(x^0,x^i); \qquad \hat{V}^\mu = \hat{V}_\mu = (\hat{V}^0, \hat{V}^i) \tag{B.2}$$

$$\hat{\partial}_0 = -i\partial_0; \qquad \hat{\partial}^i = +\partial_i = -\partial^i; \qquad \hat{\partial}^\mu = \hat{\partial}_\mu = \frac{\partial}{\partial \hat{x}^\mu} \tag{B.3}$$

$$\hat{\gamma}^0 = -i\gamma^0; \qquad \hat{\gamma}^i = -\gamma^i; \qquad \hat{\gamma}^\mu = \hat{\gamma}_\mu = (\hat{\gamma}^0, \hat{\gamma}^i) = -\hat{\gamma}_\mu^\dagger \tag{B.4}$$

$$\hat{x}\cdot\hat{y} = \hat{x}^\mu \hat{y}_\mu = -x^\mu y_\mu = -x\cdot y \tag{B.5}$$

$$\hat{\gamma}_5 = \hat{\gamma}^0\hat{\gamma}^1\hat{\gamma}^2\hat{\gamma}^3 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma_5 \tag{B.6}$$

$$\hat{\not\partial} = \hat{\gamma}\cdot\hat{\partial} = -\gamma\cdot\partial = -\gamma^\mu\partial_\mu = -\not\partial; \qquad \hat{\not V} = \hat{\gamma}\cdot\hat{V} = \gamma\cdot V = \not V \tag{B.7}$$

$$\hat{\epsilon}^{0123} = \hat{\epsilon}_{0123} = +1; \qquad \epsilon^{0123} = -\epsilon_{0123} = +1 \tag{B.8}$$

$$\hat{\epsilon}^{\mu\nu\alpha\beta}\hat{A}^\mu\hat{B}^\nu\hat{C}^\alpha\hat{D}^\beta = -i\epsilon_{\mu\nu\alpha\beta}A^\mu B^\nu C^\alpha D^\beta \tag{B.9}$$

The factor in the exponential reads

$$\exp\Big\{ i\int d^4x \bar{q} i\mathbf{D} q \Big\} = \exp\Big\{ -\int d^4\hat{x} \bar{q} i\hat{\mathbf{D}} q \Big\} \tag{B.10}$$

with the Dirac operator in Euclidean space

$$i\hat{\mathbf{D}} = i\hat{\not\partial} - \hat{\not{\mathcal{V}}} - \hat{\not{\mathcal{A}}}\hat{\gamma}_5 + \hat{\mathcal{S}} + i\hat{\gamma}_5\hat{\mathcal{P}} \tag{B.11}$$

whose hermitean conjugate is given by

$$-i\hat{\mathbf{D}}^\dagger = -i\hat{\not\partial} + \hat{\not{\mathcal{V}}} - \hat{\not{\mathcal{A}}}\hat{\gamma}_5 + \hat{\mathcal{S}} - i\hat{\gamma}_5\hat{\mathcal{P}} \tag{B.12}$$

Notice that the hermitean conjugation in Euclidean space $\hat{\mathbf{D}} \to \hat{\mathbf{D}}^\dagger$ corresponds to the operation $\mathbf{D} \to \mathbf{D}_5$ in Minkowski space (see eq. (4.2)). The fermion determinant is

$$\overline{\mathrm{Det}}(i\hat{\mathbf{D}}) = \exp(-\hat{\overline{W}}) = \overline{\mathrm{Det}}(i\mathbf{D}) = \exp(i\overline{W}) \tag{B.13}$$

The vector-additively regularized real part of the fermionic contribution to the effective action reads then

$$\mathrm{Re}\,\hat{\overline{W}} = \frac{1}{4}\mathrm{Sp}\int_0^\infty \frac{d\tau}{\tau}\Phi(\tau)\left[e^{-\tau\hat{\mathbf{D}}\hat{\mathbf{D}}^\dagger} + e^{-\tau\hat{\mathbf{D}}^\dagger\hat{\mathbf{D}}}\right] \tag{B.14}$$

Here $\Phi(\tau)$ is a generalized proper-time regularization function with $\Phi(0) = \Phi'(0) = 0$ and $\Phi(\infty) = 1$. The Pauli-Villars regularizations correspond to the choice $\Phi = \sum_i c_i \exp(-\tau\Lambda_i^2)$. The imaginary part takes the form

$$\mathrm{Im}\,\hat{\overline{W}} = \frac{i}{2}\mathrm{Sp}\left[\log(i\hat{\mathbf{D}}) - \log(-i\hat{\mathbf{D}}^\dagger)\right] \tag{B.15}$$

The variation of the determinant in such regularization involves the imaginary part of the Euclidean action only giving

$$\begin{aligned}\delta\log\overline{\mathrm{Det}}(i\hat{\mathbf{D}}) &= -i\delta\mathrm{Im}\hat{\overline{W}} = \\ &-\frac{iN_c}{4\pi^2}\hat{\epsilon}_{\mu\nu\alpha\beta}\int d^4\hat{x}\,\mathrm{tr}\Big\{\epsilon_A(x)\Big[\frac{1}{4}[i\hat{\mathcal{D}}_\mu, i\hat{\mathcal{D}}_\nu][i\hat{\mathcal{D}}_\alpha, i\hat{\mathcal{D}}_\beta] - \frac{1}{3}\hat{\mathcal{A}}_\mu\hat{\mathcal{A}}_\nu\hat{\mathcal{A}}_\alpha\hat{\mathcal{A}}_\beta \\ &-\frac{2}{3}\hat{\mathcal{A}}_\mu[i\hat{\mathcal{D}}_\nu, i\hat{\mathcal{D}}_\alpha]\hat{\mathcal{A}}_\beta - \frac{1}{6}\{[i\hat{\mathcal{D}}_\mu, i\hat{\mathcal{D}}_\nu], \hat{\mathcal{A}}_\alpha\hat{\mathcal{A}}_\beta\} + \frac{1}{3}[i\hat{\mathcal{D}}_\mu, \hat{\mathcal{A}}_\nu][i\hat{\mathcal{D}}_\alpha, \hat{\mathcal{A}}_\beta]\Big]\Big\}\end{aligned} \tag{B.16}$$

where $i\hat{\mathcal{D}}_\mu = i\hat{\partial}_\mu - \hat{\mathcal{V}}_\mu$ has been defined.

Appendix C. Results for Hedgehog Profiles

Here are the expressions for the modification of the computed quantities and the total results for self-consistent fields not given in the main text.

C.1 Modification of nucleon observables

Mean field energy

$$\begin{aligned}\Delta E[\omega,\rho,A] = -\frac{N_c}{12\pi^2}g_\rho^2 g_\omega \int_0^\infty 4\pi r^2 dr\omega\Big\{2\rho(A_S' - \frac{1}{3}A_T' + \frac{1}{r}A_T) \\ + 4(\rho' + \frac{2}{r}\rho)(A_S - \frac{1}{3}A_T) + \frac{3g_\rho}{4}(A_S + \frac{2}{3}A_T)\Big[(A_S - \frac{1}{3}A_T)^2 + \rho^2\Big]\Big\}\end{aligned} \tag{C.1}$$

Isoscalar baryon density

$$\Delta J_0^B = \frac{N_c}{4\pi^2}g_\rho^2\frac{1}{r^2}\frac{d}{dr}\Big\{r^2(A_S - \frac{1}{3}A_T)\rho\Big\} \tag{C.2}$$

Vector-isovector current

$$(\Delta J^V)_a^i = -\epsilon_{iak}\hat{x}_k\frac{N_c}{24\pi^2}g_\rho g_\omega\Big\{3\Big[(A_S - \frac{1}{3}A_T)\omega\Big]' - \frac{3A_T\omega}{r} - 2g_\rho\omega\rho(A_S + \frac{2}{3}A_T)\Big\} \tag{C.3}$$

Axial isovector current

$$\begin{aligned}(\Delta J^A)_a^i = \frac{N_c}{24\pi^2}g_\rho g_\omega\Big\{\delta_{ia}\Big[3\omega'\rho - 3\omega(\rho' + \frac{\rho}{r}) - 2g_\rho\omega(A_S - \frac{1}{3}A_T)(A_S + \frac{2}{3}A_T)\Big] \\ + \hat{x}_i\hat{x}_a\Big[-3\omega'\rho + 3\omega(\rho' - \frac{\rho}{r}) - 2g_\rho\omega\Big(\rho^2 - (A_S - \frac{1}{3}A_T)A_T\Big)\Big]\Big\}\end{aligned} \tag{C.4}$$

Isoscalar radius

$$\langle\Delta r^2\rangle_N^{I=0} = -\frac{N_c}{2\pi^2}\int_0^\infty 4\pi r^3 dr(A_S - \frac{1}{3}A_T)\rho \tag{C.5}$$

Axial coupling constant

$$\Delta g_A^N = -\frac{N_c}{24\pi^2}g_\rho g_\omega\frac{2}{3}\int_0^\infty 4\pi r^2 dr 2\omega\Big[2(\rho' + \frac{2\rho}{r}) + \frac{1}{3}g_\rho\rho^2 + g_\rho(A_S - \frac{1}{3}A_T)(A_S + \frac{1}{3}A_T)\Big] \tag{C.6}$$

C.2 Self-consistent nucleon observables

Isoscalar baryon density

$$\begin{aligned}(J^B)^0 = -\frac{m_\omega^2}{g_\omega}\omega + \frac{N_c}{12\pi^2}g_\rho^2\Big\{\rho\Big(A_S' - \frac{1}{3}A_T' - \frac{2A_T}{r}\Big) - (A_S - \frac{1}{3}A_T)(\rho' + \frac{2}{r}\rho) \\ - \frac{3}{4}g_\rho(A_S + \frac{2}{3}A_T)\Big[(A_S - \frac{1}{3}A_T)^2 + \rho^2\Big]\Big\}\end{aligned} \tag{C.7}$$

Isospin current

$$
\begin{aligned}
(J^V)^i_a = \epsilon_{iak}\hat{x}_k\Big\{\frac{m_\rho^2}{g_\rho}\rho + \frac{N_c}{24\pi^2}g_\rho g_\omega\Big[&(A_S - \frac{1}{3}A_T)\omega' \\
&- (A'_S - \frac{1}{3}A'_T)\omega + \frac{A_T\omega}{r} + \frac{1}{2}g_\rho\omega\rho(A_S + \frac{2}{3}A_T)\Big]\Big\}
\end{aligned}
\tag{C.8}
$$

Axial current

$$
\begin{aligned}
(J^A)^i_a = &-\delta_{ia}\Big\{\frac{m_\rho^2}{g_\rho}(A_S - \frac{A_T}{3}) + \frac{N_c}{24\pi^2}g_\rho g_\omega\Big[-\omega'\rho + \omega\rho' + \frac{\rho\omega}{r} + \frac{1}{2}g_\rho\omega(A_S - \frac{A_T}{3})(A_S + \frac{2A_T}{3})\Big\} \\
&- \hat{x}_i\hat{x}_a\Big\{\frac{m_\rho^2}{g_\rho}A_T + \frac{N_c}{24\pi^2}g_\rho g_\omega\Big[\omega'\rho - \omega\rho' + \frac{\rho\omega}{r} - \frac{1}{2}g_\rho\omega\Big((A_S - \frac{1}{3}A_T)A_T - \rho^2\Big)\Big]\Big\}
\end{aligned}
\tag{C.9}
$$

Axial coupling constant

$$
g_A^N = -\frac{2}{3}\int d^3x\Big\{\frac{m_\rho^2}{g_\rho}A_S + \frac{N_c}{36\pi^2}g_\rho g_\omega\Big[-\omega'\rho + \omega\rho' + \frac{2\omega\rho}{r} + \frac{3}{4}g_\rho(A_S^2 - \frac{1}{9}A_T^2 + \frac{1}{3}\rho^2)\Big]\Big\}
\tag{C.10}
$$

Appendix D. Five dimensional expression for the gauged Wess-Zumino term

In this appendix we give an alternative form for the gauged Wess-Zumino term both in its right-left and vector versions, $\Gamma^{\rm RL}_{\rm WZ}(U, v_R, v_L)$ and $\Gamma^{V}_{\rm WZ}(U, v, a)$ respectively, as five-dimensional integrals. The interesting feature of these formulas is that, oposite to the more conventional form, (see eq. (6.1)), the chiral transformation properties become more evident.

The topological Wess-Zumino term reads [4]

$$\Gamma_{\rm WZ}[U] = -\frac{iN_c}{48\pi^2}\int_{D_5} {\rm tr}\left[\frac{1}{10}U_R^5\right] \tag{D.1}$$

where D_5 represents a five-dimensional manifold with boundary the compactified four-dimensional space-time, $U_R = U^\dagger dU$ is a 1-form living in D_5 and $U(x,s)$ is an interpolating unitary flavour field, $U^\dagger U = 1$, satisfying $U(x,1) = U(x)$ and $U(x,0)$ a constant matrix. The following two observations will be used below. Since the integrand is a closed form (i.e. locally exact) the result does not depend (modulo homotopy classes) on the particular choice of D_5. Therefore the action is well-defined modulo $2\pi i$. In addition, in this form the topological Wess-Zumino term is manifestly invariant under global chiral transformations but not under local ones. As pointed out in ref. [4] the traditional minimal substitution method cannot be applied in its standard form since the topological action is non local. In fact the existence of the anomaly prevents a full chiral gauging. Thus one has to resort to other methods [38]. For instance, eq. (6.1) can be obtained by trial and error gauging.

An alternative way to obtain the gauged Wess-Zumino term is the following. We consider the minimal substitution rule directly in five dimensions, i.e. we make $U_R \to R$ with R the natural five-dimensional extension of eq. (9.4). This requires introducing five-dimensional gauge fields as well. Clearly $R(x,s)$ is chirally covariant and hence the new action is formally chirally invariant. However, since the integrand R^5 is not a closed 5-form the action is no longer independent of the choice of D_5 modulo $2\pi i$. To reestablish the one-valuedness of the action one has to supplement the integrand with additional terms satisfying the following two conditions: i) the total sum has to be closed and ii) the new terms have to vanish in the absence of gauge fields. This uniquely determines the result. For the right-left representation we find

$$\begin{aligned}\Gamma^{\rm RL}_{\rm WZ}[U, v_R, v_L] = -\frac{iN_c}{48\pi^2}\int_{D_5} {\rm tr}\Big[&\frac{1}{10}R^5 - 2RF_R^2 + iR^3F_R - RF_RU^\dagger F_LU \\ &+2iv_RF_R^2 - v_R^3F_R - \frac{i}{5}v_R^5\Big] - {\rm p.c.}\end{aligned} \tag{D.2}$$

One should mention that the usual integration by parts is precluded in this formula since D_5 has a boundary. One can check that the difference between the gauged and the non gauged Wess-Zumino terms is in fact the five-dimensional integral of an exact 5-form which is the differential of the corresponding four-dimensional piece in (6.1). On the other hand, the vector representation, again obtained by Bardeen's subtraction, reads

$$\begin{aligned}\Gamma^{\rm VA}_{\rm WZ}[U, v, a] = -\frac{iN_c}{48\pi^2}\int_{D_5} {\rm tr}\Big[&\frac{1}{10}R^5 - 2RF_R^2 + iR^3F_R - RF_RU^\dagger F_LU \\ &+2iaF_RF_L + 4iaF_R^2 - 8a^3F_R - i\frac{16}{5}a^5\Big] - {\rm p.c.}\end{aligned} \tag{D.3}$$

Besides the fact that the last two expressions are more compact than the corresponding ones, eqs. (6.1) and (6.6), the chirally breaking terms are manifestly polynomial and hence the anomaly. Let us remark that these polynomial terms do not form an exact differential in five-dimensions and hence they cannot be subtracted by adding four-dimensional polynomial counterterms. In other words the anomaly cannot be removed.

Appendix E. CP-violating currents

As already mentioned in section 6, the counterterms are unique if CP-invariance is invoked. The only possible CP-violating counterterms are given by

$$\Gamma_{\rm ct}^{\rm CP} = -ic\frac{N_c}{48\pi^2}\int {\rm tr}\left[V_R^2 F_R - V_L^2 F_L\right] \tag{E.1}$$

with c an arbitrary real constant as implied by CPT. The contributions to the currents are

$$\int {\rm tr}\left(vJ_V^{\rm CP} + aJ_A^{\rm CP}\right) = -ic\frac{N_c}{24\pi^2}\int {\rm tr}\left(dv(VA+AV) + da(V^2+A^2)\right) \tag{E.2}$$

In the two flavour case we have

$$\begin{aligned}
(J^B)_\mu^{\rm CP} &= 0 \\
(\vec{J}^V)_\mu^{\rm CP} &= \frac{N_c}{48\pi^2} c g_\rho^2 \epsilon_{\mu\nu\alpha\beta}\partial^\nu(\vec{\rho}^\alpha \wedge \vec{A}^\beta) \\
(\vec{J}^A)_\mu^{\rm CP} &= \frac{N_c}{96\pi^2} c g_\rho^2 \epsilon_{\mu\nu\alpha\beta}\partial^\nu(\vec{\rho}^\alpha \wedge \vec{\rho}^\beta + \vec{A}^\alpha \wedge \vec{A}^\beta)
\end{aligned} \tag{E.3}$$

For hedgehog profiles all CP-violating currents vanish except the time component of the axial current

$$(\vec{J}_A)_0 = \frac{N_c}{24\pi^2} c g_\rho^2 \hat{x}_a \Big\{\rho(\rho' + \frac{\rho}{r}) + (A_S - \frac{A_T}{3})(A_S' - \frac{A_T'}{3} - \frac{A_T}{r})\Big\} \tag{E.4}$$

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